Analysis of linear polarization basing on the single Stokes parameter

G. V. Lipunova

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Abstract

One MASTER telescope is equipped with two orthogonal polarizers. It happens sometimes that a single telescope observes a transient source. We investgate what information about the source's linear polarization P_L can be learned from the single Stokes' parameter. For the case of zero Stokes parameter, dependences of the 1σ and 2σ P_L upper limits on the Stokes parameter's uncertainty σD are found. Different values of observed Stokes parameter correspond to different dependences $max(P_L)$ vs. σD. They can be calculated by the method proposed.

1 Limit on the degree of the linear polarization

When observing with only two perpendicular polaroids, just one Stokes parameter can be inferred. This Stokes parameter is the lower limit on the degree of the linear polarization P_L. The polarization angle cannot be defined.

Let I_1, I_2 be observable fluxes in two perpendicular polaroids. We derive value

$$D = \frac{I_1 - I_2}{I_1 + I_2}\,.$$

If I is the total flux (the value proportional to counts) from the source, then

$$I_1 = I\,P_L\,\cos^2\theta \qquad I_2 = I\,P_L\,\sin^2\theta$$

and one Stokes parameter is

$$D = P_L\,\cos 2\theta\,.$$

If errors σI of I_1 and I_2 are normally distributed, $I_1 = I_2$, then approximately

$$\frac{\sigma D}{D} = \frac{1}{\sqrt{2}}\,\frac{\sigma I}{I}$$

(But see Simmons J. F. L. & Stewart B. G. 1985, A&A, 142, 100).

2 Allowed values of P_L from D, zero noise

If a source with the degree of linear polarization P_L is observed at some polarization angle θ, the noise-free Stokes parameter is as follows:

$$D(\theta) = P_L \times \cos(2\,\theta)$$

We can find the range of allowable angles that corresponds to some variation

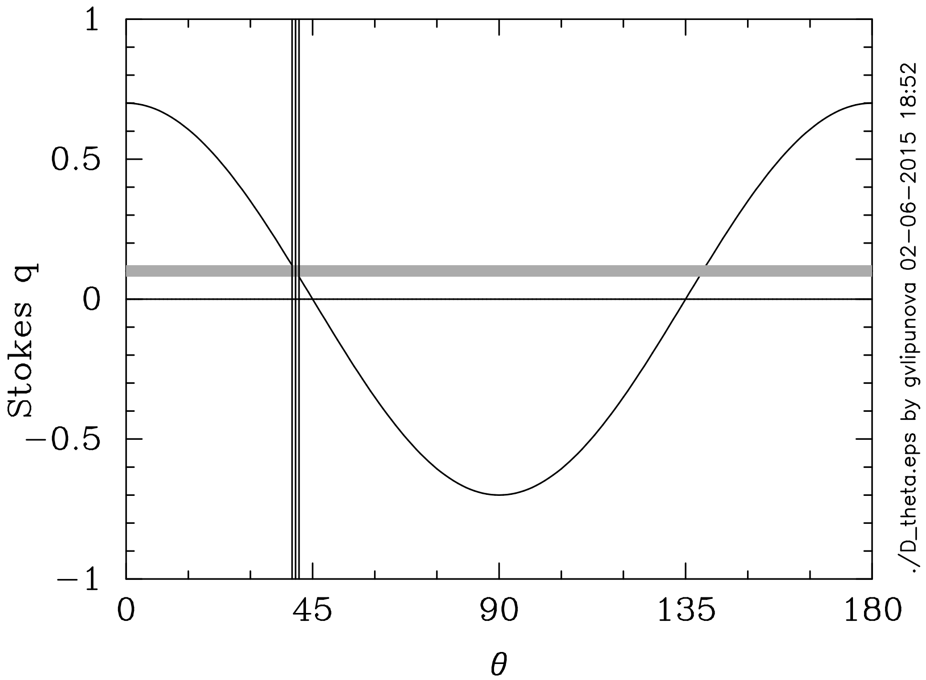

Figure 1: Dimensionless Stokes parameter versus polarization angle. The grey band corresponds to an observed value, $D_o \pm \sigma D = 0.1 \pm 0.02$. Three vertical lines correspond to three values of angle, from left to right: $\theta_o - \delta_1$, θ_o, and $\theta_o + \delta_2$ – see relations (1).

of D:

$$D_o = P_L \times \cos(2\,\theta_o)$$

$$D_o + \sigma D = P_L \times \cos(2\,(\theta_o - \delta_1)) \qquad D_o - \sigma D = P_L \times \cos(2\,(\theta_o + \delta_2)) \qquad (1)$$

Probability for the polarizer's direction to occur in the corresponding range over angles is

$$(\delta_1 + \delta_2)\,2/\pi \qquad (2)$$

as can be seen from Fig. 1. For a set of values P_L, D_o, and σD, system (1) can be solved to find θ_o, δ_1, and δ_2. Values (2) are shown by the green lines in Figs.2–8, designated as 'Probability'.

If $D_o + \sigma D > P_L$, δ_1 or δ_2 are cut accordingly to provide $\cos = 1$. Thus, for values $P_L = 0$ non-zero D cannot be obtained in the case of zero noise. This is a shortcoming of such deterministic method. To address possible errors of observed D, we perform simulations using the Monte-Carlo method.

3 Application of the Bayes' theorem

Let X and Y are the continous random variables. Consider events $X = x$ and $Y = y$, where x and y are some numerical values. Consider the the Bayes' theorem, formulated in terms of the probability densities f_X and f_Y,

$$f_X(x|Y=y) = \frac{f_Y(y|X=x)\, f_X(x)}{f_Y(y)} \tag{3}$$

Let X be all possible values of the degree of linear polarization: $X = P_L$ and $\in [0,1]$. Then $x = P_L^*$ is the degree of the linear polarization of the source.

Let Y be all possible values of the dimesionless Stokes parameter $Y = D$, which can be observed, and $Y \in [-1,1]$. For the specific data, we derive the dimesionless Stokes parameter D_o.

According to (3), the probability that the source has the linear degree of polarization P_L^* if we observe value D_o is:

$$f_{P_L}(P_L^*|D=D_o) = \frac{f_D(D=D_o|P_L=P_L^*) \times f_{P_L}(P_L^*)}{f_D(D_o)} \tag{4}$$

It is be more practical to find the following probability:

$$f_{P_L}(P_L \le P_L^*|D=D_o) = \frac{f_D(D=D_o|P_L \le P_L^*) \times f_{P_L}(P_L \le P_L^*)}{f_D(D_o)} \tag{5}$$

It is assumed that the probability of a source to have specific P_L^* is uniform:

$$f_{P_L}(P_L^*) = 1\,. \tag{6}$$

In its turn, the probability density of observing certain D_o from a source with some polarization less than P_L^* is the limit of the ratio of probability dP to measure D inside some small interval $[D_o - \mathrm{d}D..D_o + \mathrm{d}D]$ to the size of the interval:

$$f_D(D=D_o|P_L \le P_L^*) = \frac{dP(D \in [D_o - \mathrm{d}D..D_o + \mathrm{d}D] \Big| P_L \le P_L^*)}{2\,\mathrm{d}D} \tag{7}$$

Similarly,

$$f_D(D_o) = \frac{dP(D \in [D_o - \mathrm{d}D..D_o + \mathrm{d}D])}{2\,\mathrm{d}D} \tag{8}$$

3.1 Monte-Carlo simulations

Performing Monte-Carlo simulations, we can derive probability density (7) as follows. Let us generate n sources with identical P_L^* and different polarization angles θ, uniformly distributed over interval $[0..\pi]$. Consequently, we obtain diffrent values of $D = P_L \cos 2\theta$, which would be observed from such sources in the case of zero noise. To take into account the noise (experimental random errors) we shift each D by a random value distributed as $N(0, \sigma D)$ (normal distribution with zero mean and standard deviation σD) and obtain D_{sh}. Hence we presume that the absolute error of observed D is distributed normally.

We vary value P_L^* from 0 to 1 at equal steps, each time generating n points, and count the number of the following events:

N_{tot} — total number of points.

N_A — when $P_L \le P_L^*$.

N_B — when $|D_{sh} - D_o| \le \mathrm{d}D$.

$n_B(P_L)$ — when $|D_{sh} - D_o| \le \mathrm{d}D$ for each value P_L.

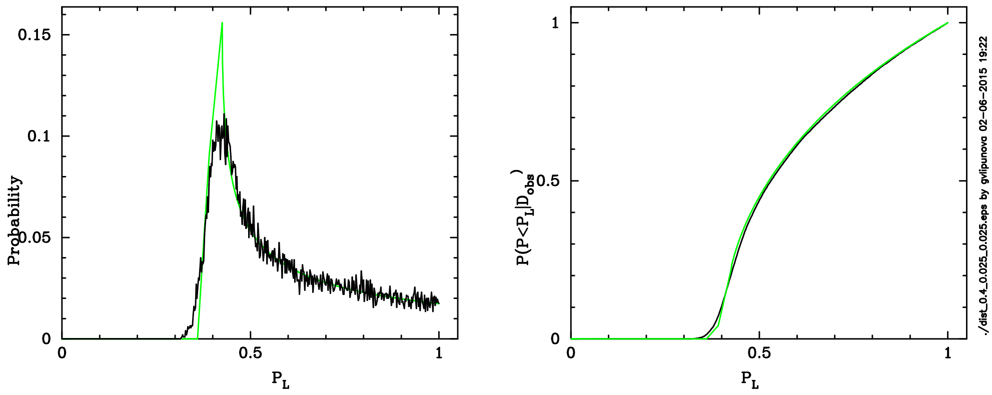

Figure 2: $D_o = 0.4$, $\sigma D = 0.025$, $dD = 0.025$.

N_{BA} — when $|D_{sh} - D_o| \le \mathrm{d}D$ and $P_L \le P_L^*$ at the same time.

Value $\mathrm{d}D$ can be chosen arbitrarily but it should be of order of σD.

The simulation consists of 501 steps over P_L with parameters $n = 2000$, $\mathrm{d}D = \sigma D$ or $\mathrm{d}D = 0.5 \times \sigma D$. The total numer of events is $N_{tot} = 1002000$.

We understand that

$$P(P_L \le P_L^*) \equiv P_L^* = N_A/N_{tot} \qquad \sim P(A) \tag{9}$$

$$f_D(D_o) = N_B/N_{tot} \qquad \sim P(B) \tag{10}$$

$$f_D(D = D_o|P_L \le P_L^*) = N_{BA}/N_A \qquad \sim P(B|A) \tag{11}$$

Furthermore, the probability that a source with P_L gives observed D in the interval $[D_o - dD ... D_o + dD]$ is

$$P(P_L, D_o, \sigma D, dD) = n_B/n \,. \tag{12}$$

Evidently, the last function depends on the value of dD. This function, designated as 'Probability', is shown by black curves in Figs. 2-8 and is to be compared with the result obtained by (2) shown by the green lines.

Thus, the value, which we seek,

$$f_{P_L}(P_L \le P_L^*|D = D_o)$$

can be found as

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} = \frac{N_{BA}}{N_B}\,. \tag{13}$$

The last value should be normalized by the maximum value, giving $f_{P_L}(P_L \le 1|D = D_o) = 1$ at the extreme case. The result is shown in Figs. 2-9 in the correspondingly named panels. It can be also calculated as the cumulative sum of (2) (green lines), also normalized by its maximum value.

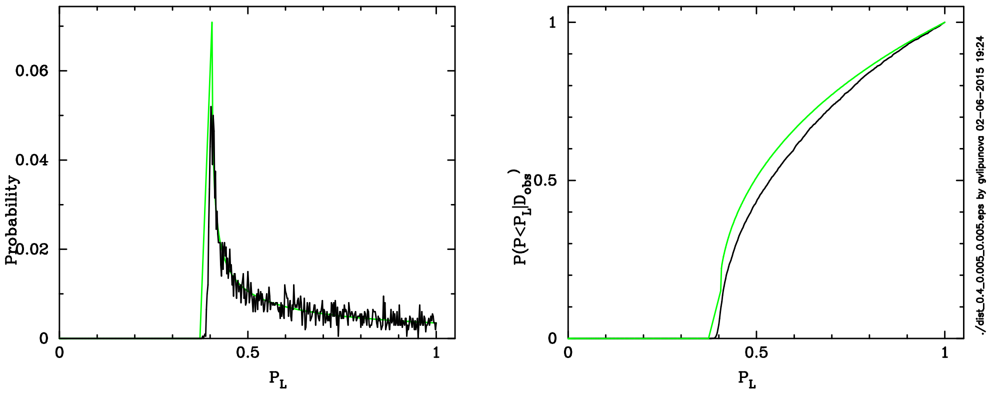

Figure 3: $D_o = 0.4$, $\sigma D = 0.005$, $dD = 0.005$.

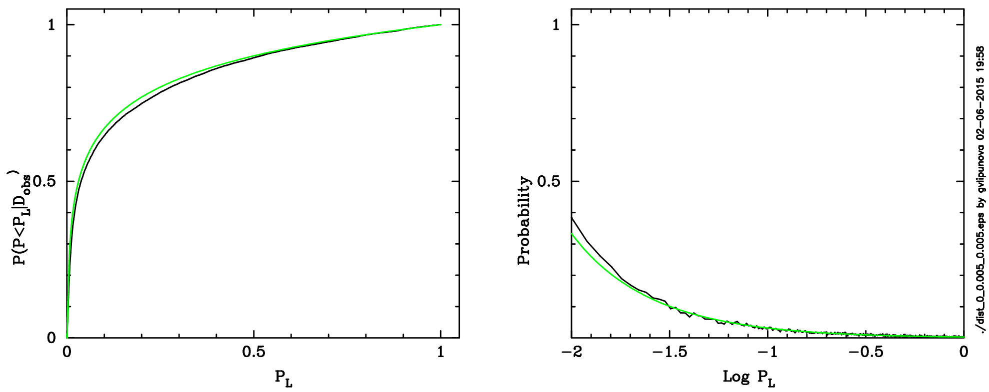

Figure 4: $D_o = 0$, $\sigma D = 0.005$, $dD = 0.005$.

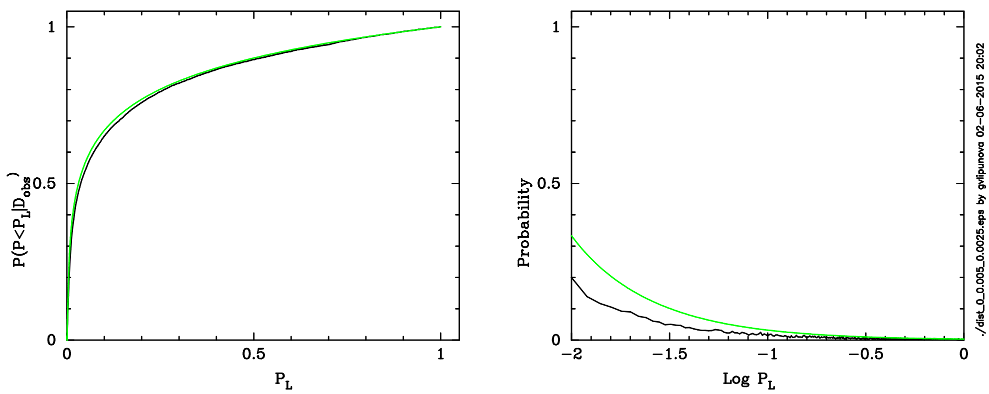

Figure 5: $D_o = 0$, $\sigma D = 0.005$, $dD = 0.0025$.

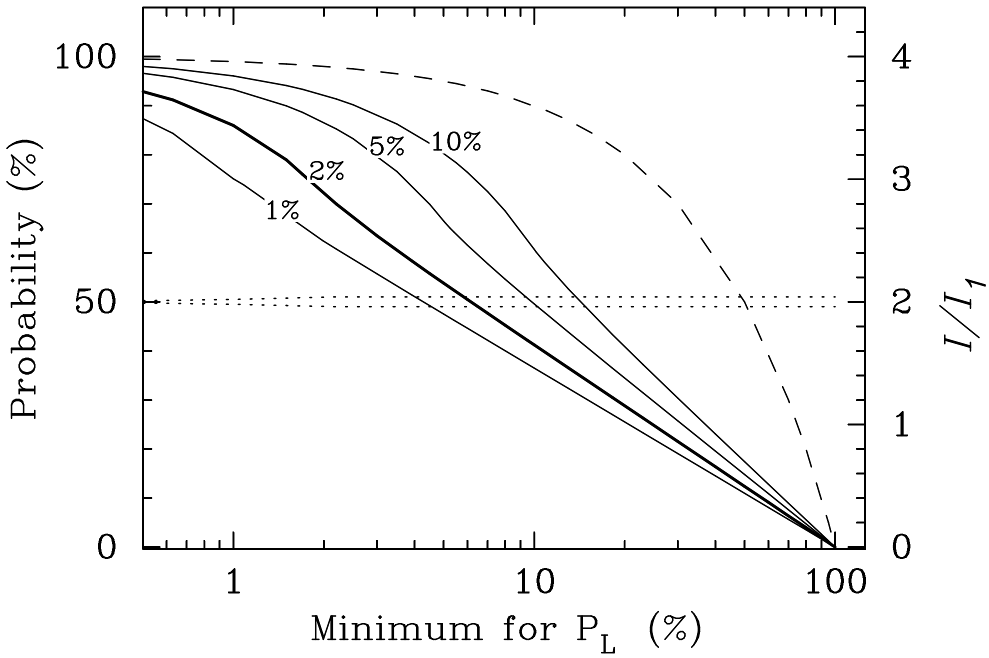

Figure 6: Previous results for the probability calculated by (2). Values σD are shown for the corresponding curves. The 2%-curve agrees with the green curves in Figs. 7 and 8 (the left upper panel). Figure is from Gorbovskoy et al. (2012)

3.2 Comparison to previous results

In Gorbovskoy et al (2012) a result using formula (2) for $D_o = 0$ was reported, which we reproduce here. Fig. 6 shows the probability of the degree of linear polarization to be greater than the value on the horizontal axis. The case of observed $D_o = 0$ is considered (zero Stokes' parameter). Different curves are plotted for different σD. In Figs. 7 and 8 we show that those results are consistent with the present ones (the green curves in panels for $P(P > P_L | D_{\rm obs})$).

4 Upper limits on P_L for different σD and different confidence levels

The practical interest is to provide a limit on P_L-value of a source basing on the observed D_o.

In Fig. 10, results for $D_o = 0$ using (2) are presented. We calculate the same dependences via Monte-Carlo simulations described above (Fig. 11).

The upper limits can be calculated for other values of D_o. An example, corresponding to the case of GRB 140801, is plotted in Fig. 9.

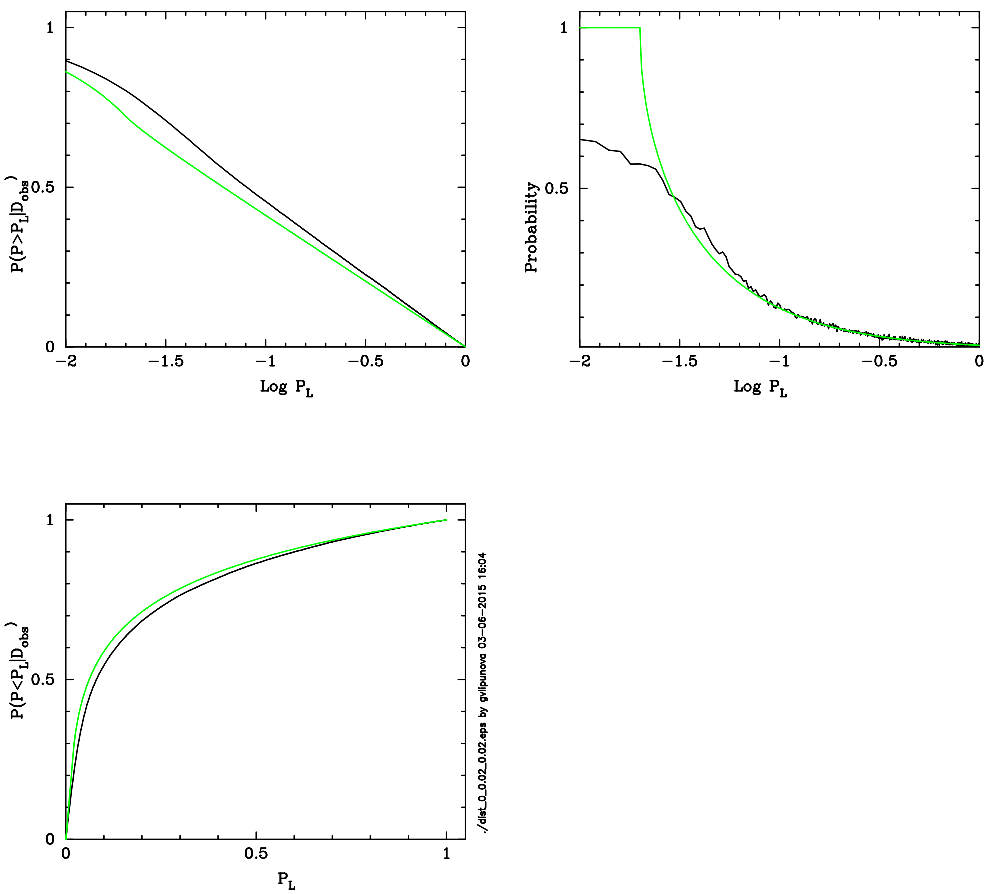

Figure 7: $D_o = 0$, $\sigma D = 0.02$, $\Delta D = 0.02$.

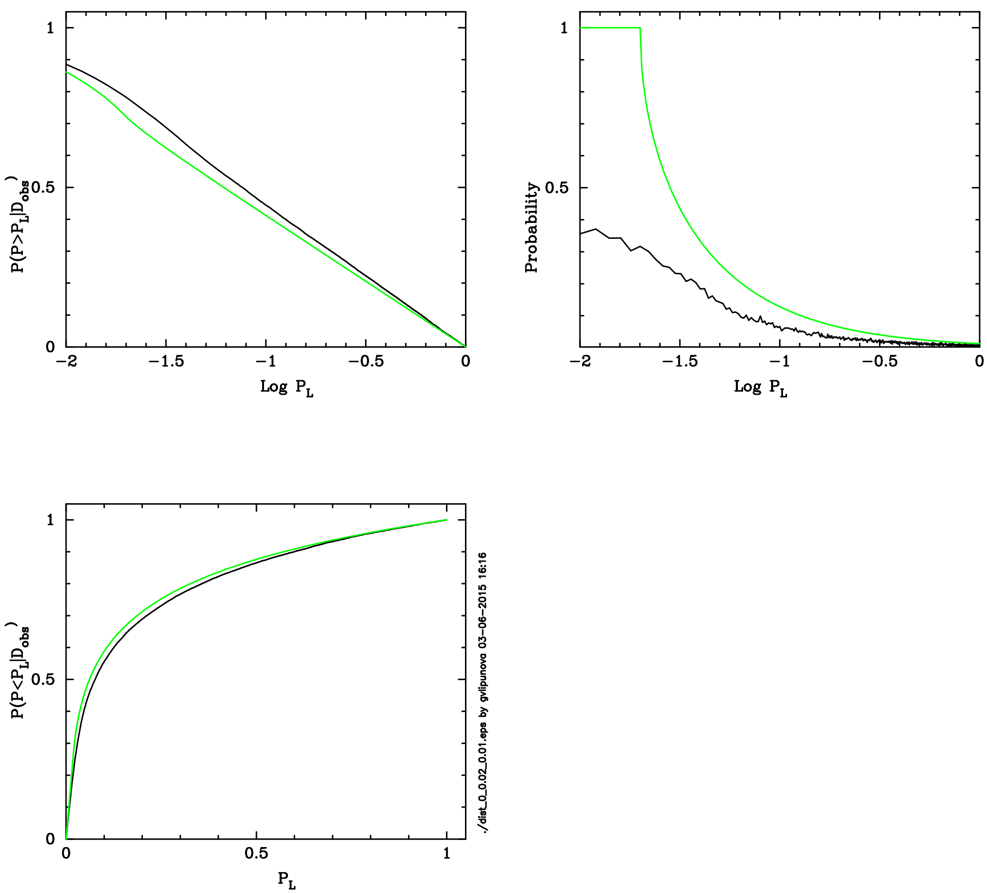

Figure 8: $D_o = 0$, $\sigma D = 0.02$, $\Delta D = 0.01$.

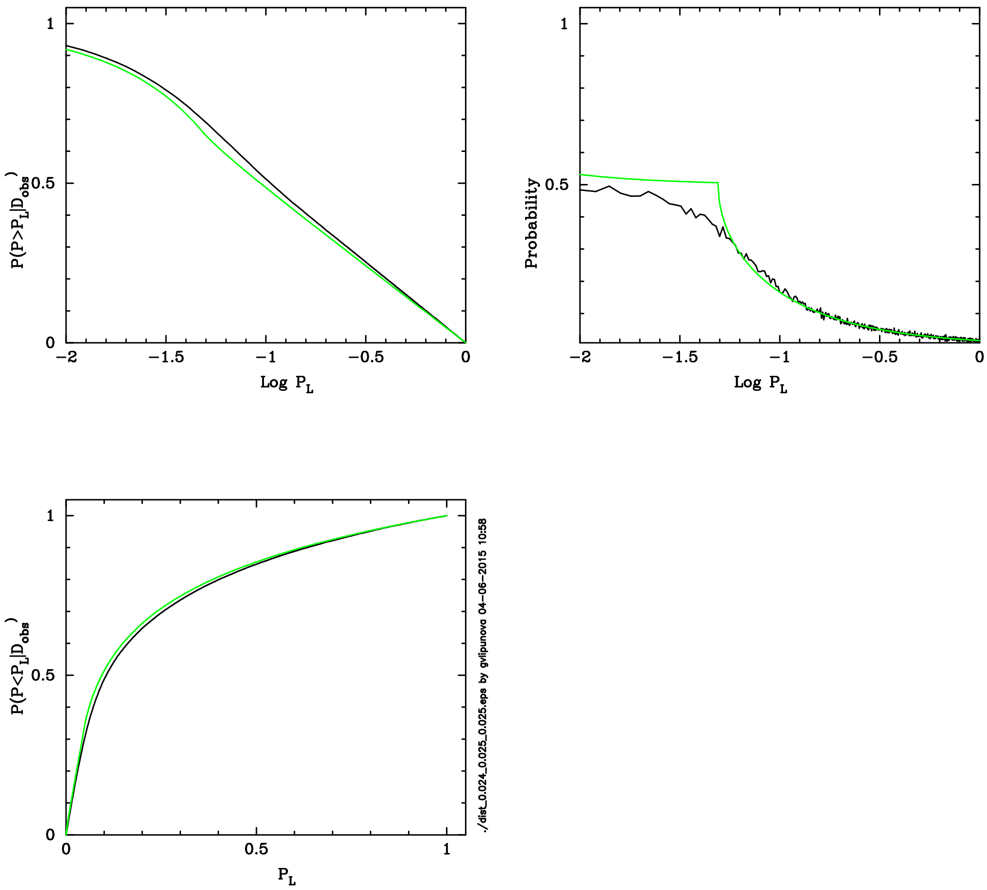

Figure 9: $D_o = 0.024$, $\sigma D = 0.025$, $\Delta D = 0.025$. 1-σ upper limit on P_L is $\sim 24\%$ and 2-σ upper limit on P_L is $\sim 81\%$. These are slightly higher than the limits for $D_o = 0$ and $\sigma D = 0.025$, $\sim 21\%$ and $\sim 80\%$, which can be found from Fig. 11. The case of GRB 140801.

Case: Stokes parameter is less than relative accuracy σ

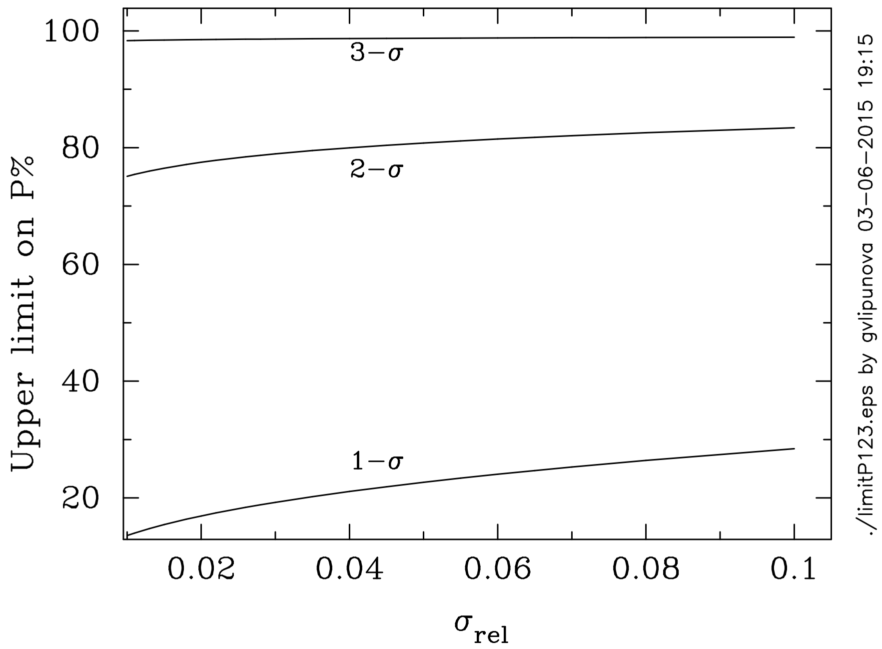

Figure 10: For observed $D_o = 0$, the value on the vertical axis represents the upper limit on then degree of linear polarization versus $\sigma D \equiv \sigma_{rel}$. Three curves correspond to different confidence levels, designated for each curve.

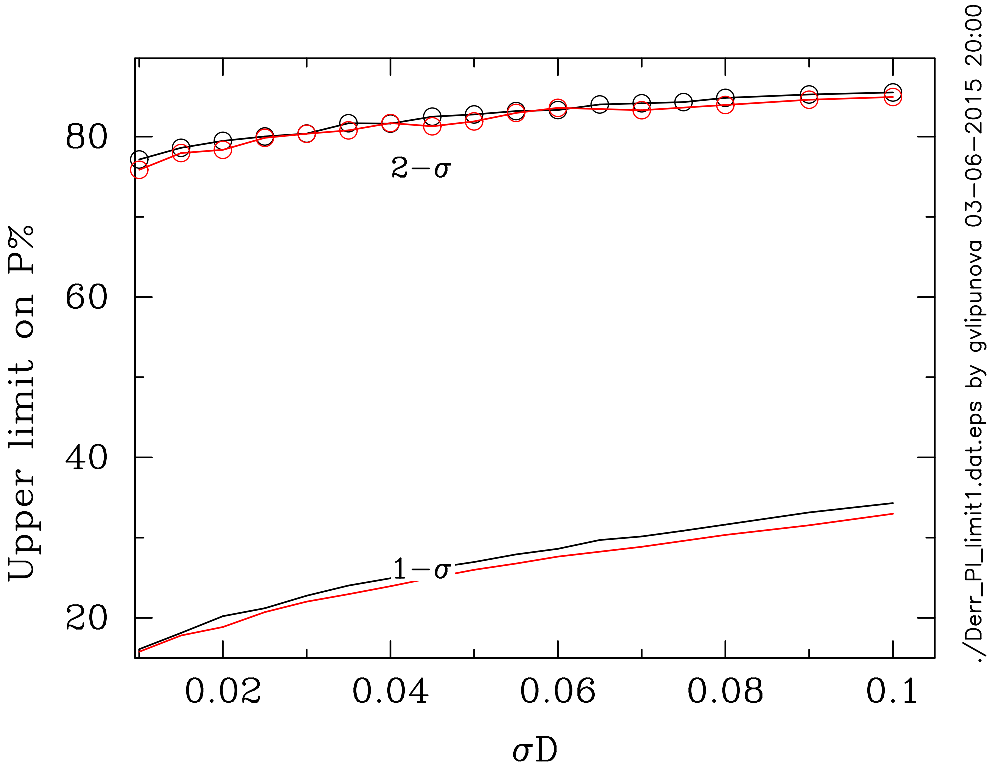

Figure 11: Results of Monte-Carlo simulations for $D_o = 0$, $dD = \sigma D$ (black) and $dD = 0.5\,\sigma D$ (red). The value on the vertical axis represents the upper limit on then degree of linear polarization versus σD. Two curves correspond to different confidence levels, designated for each curve.